Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #YY, 13 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Line Shellings of Bergman Fans

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Abstract. Given a matroid M and a building set B on its lattice of flats, we prove that the associated nested set complex N is shellable. This generalizes a classical result of Björner that the order complex of the lattice of flats of a matroid is shellable (the case when B is the maximum builiding set), and strengthens a result of Feichtner-Müller that N is Cohen-Macaulay for arbitrary B. Our approach is geometric in nature utilizing the Bergman fan $\Sigma_{M,B}$, and is inspired by Bruggesser and Mani's line shellings of polytopes. We prove that, given a normal complex P for $\Sigma_{M,B}$, as introduced by Ross and the third author, and a lexicographic weight vector ω, the order of the vertices of P induced by ω is a shelling order for N. As special cases, our result implies shellability of the polymatroid Bergman fan, and gives new proofs of shellability for the fine and augmented Bergman fans. We compare our shelling order with Björner's order for the order complex of the lattice of flats.

Keywords: matroid, Bergman fan, nested set, building set, shelling

1 Main Results

Let M be a loopless matroid on a finite ground set E and $\mathcal{L}$ the lattice of flats of M. A ***building set*** $B \subseteq \mathcal{L} \setminus \{\emptyset\}$ is a subset of the flats of $\mathcal{L}$ such that for all $X \in \mathcal{L} \setminus \{\emptyset\}$, there is an isomorphism of posets

$$\left(\prod_{Y \in \max(B_{\leq X})} [\emptyset, Y] \right) \cong [\emptyset, X] \, ,$$

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where $\max(B_{\leq X})$ denotes the containment-maximal elements of B that lie below X.[1] Note that we do not require E to be an element of the building set; see Section 2 an example of a building set that does not contain E. A subset $N \subseteq B$ is ***nested*** if for every antichain $X_1, \ldots, X_\ell \in N$ with $\ell \geq 2$, the join of that antichain is in $\mathcal{L} \setminus B$. That is, $\bigvee_{i=1}^{\ell} X_i \in \mathcal{L} \setminus B$. The ***nested set complex*** of a building set B in $\mathcal{L}$ is the simplicial complex whose faces are the nested sets $N \subseteq B$. We will use $\Delta(\mathcal{L}, B)$ to denote the nested set complex of $\mathcal{L}$ and B.

The nested set complex of the ***maximal building set*** $B^{\max} = \mathcal{L} \setminus \{\emptyset\}$ is the order complex of $\mathcal{L} \setminus \{E, \emptyset\}$. The combinatorial topology of such complexes is rich and well studied; see [6, Sections 7.6 and 7.9] and the references within. In particular, Björner proved that the complex $\Delta(\mathcal{L}, B^{\max})$ is ***shellable*** [5, Theorem 6.1]. A simplicial complex Δ is shellable if there is an ordering $N_1 < \cdots < N_k$ of the inclusion maximal faces of Δ such that, for all i, $N_i \cap \bigcup_{j<i} N_j$ is a pure complex of codimension one. Our main theorem extends Björner's result to arbitrary building sets.

Theorem 1. *Let M be a matroid with lattice of flats $\mathcal{L}$. For any choice of building set $B \subseteq \mathcal{L}$, the nested set complex $\Delta(\mathcal{L}, B)$ is shellable.*

Let $F_1, \ldots, F_m \in B$ be the containment maximal flats of the building set B and set

$$L_B = \operatorname{span}_{\mathbb{R}}\{e_{F_1}, \ldots, e_{F_m}\} \qquad \text{where } e_F = \sum_{i \in F} e_i \,,$$

and e_i denotes the ith standard basis vector. For each nested set $N \in \Delta(\mathcal{L}, B)$ containing $F_1, \ldots, F_m$, define the cone $\sigma_N \subseteq \mathbb{R}^E$ by

$$\sigma_N = L_B + \operatorname{cone}\left(e_X \mid X \in N\right) .$$

The ***Bergman fan*** $\Sigma_{\mathcal{L},B}$ of the pair $(\mathcal{L}, B)$ is the fan in $\mathbb{R}^E$ with cones σ_N for each nested set N containing all elements of $\max(B) = \{F_1, \ldots, F_m\}$. The Bergman fan shares much of the combinatorics of the nested set complex. That is, a geometric realization of the nested set complex triangulates the Bergman complex; see [15, Theorem 4.1]. In particular, the maximal cones of $\Sigma_{\mathcal{L},B}$ are indexed by the facets of $\Delta(\mathcal{L}, B)$, i.e. inclusion maximal nested sets N. We refer to a ***shelling of the Bergman fan*** as an ordering of the maximal cones of $\Sigma_{\mathcal{L},B}$ such that the induced order on the facets of $\Delta(\mathcal{L}, B)$ is a shelling order.

In the case of the maximal building set, the Bergman fan $\Sigma_{\mathcal{L},B^{\max}}$ was first defined by [1] in their study of the topology of tropical linear spaces $\operatorname{trop}(M)$. There, $\Sigma_{\mathcal{L},B^{\max}}$ is

[1] An equivalent definition is given by the first author and Danner in [2, Proposition 2.11]. Their result says that $B \subseteq \mathcal{L} \setminus \{\hat{0}\}$ is a building set if and only if the atoms of $\mathcal{L}$ are in B and for all $X, Y \in B$ with $X \vee Y \neq \hat{0}$, we have $X \wedge Y \in B$. This second version may be more familiar to readers who study *nestohedra*.

referred to as the ***fine subdivision*** and Björner's shelling of $\Delta(\mathcal{L}, B^{\max})$ lets them determine the topology of $\mathrm{trop}(M)$. Our situation proceeds in the reverse order, we want to find a shelling of $\Delta(\mathcal{L}, B)$ and we will exploit its connections with $\Sigma_{\mathcal{L},B}$ in order to do so.

Note that most of the time, the Bergman fan is not complete, i.e., it is not true that every point in $\mathbb{R}^E$ sits inside some cone of the Bergman fan. If it was complete, there would be some (possibly not unique) polytope P for which the Bergman fan was the normal fan of P. The existence of "line shellings" of polytopes [21, Section 8.2] would then immediately imply that the Bergman fans were shellable. The work of the third author and Ross provides an alternative for this missing polytope. Given a Bergman fan $\Sigma_{\mathcal{L},B}$, they produce a compact polytopal complex $\mathcal{N}_{\mathcal{L},B}$– called a normal complex (defined in Section 3)– which we can use to study Bergman fans of matroids.

In order to prove our main theorem, we sweep a hyperplane through this normal complex in a way that produces a special linear order on the vertices of this normal complex. Then we prove that this special linear order on the vertices gives a shelling order on $\Sigma_{\mathcal{L},B}$, immediately implying Theorem 1.

Combining Theorem 1 with results from Crowley-Huh-Larson-Simpson-Wang [11], Braden-Huh-Proudfoot-Wang [7, Proposition 2.3], and an observation from Eur as described in [18, p.1843] (see also [16, Remark 3.13]) we obtain the following corollary, parts of which give new proofs[2] of [10, Theorem 1.1] and [13, Theorem 3.1] (this second result uses Trappmann–Ziegler [20] to construct the shelling order for the complete graph matroid).

Corollary 2. *The following objects are shellable:*

- *The Bergman fan and nested set complex of a polymatroid.*
- *Augmented Bergman fans of matroids*
- *The boundary complex of $\overline{M_{0,n}}$ (via the braid matroid, whose lattice of flats is the partition lattice, and the minimum building set)*

In the remainder of this extended abstract, we will define normal complexes (see Section 3) and then sketch our proof of Theorem 1 (see Section 5).

Remark 3. A recent preprint from Balla–Joswig–Weis uses line shellings to show that tropical hypersurfaces are shellable [3]. Our results do not seem to imply theirs, nor do thiers imply ours. Both works use line shellings to prove shellability of non-complete fans (a technique which– to the best of our knowledge– is unique to these two works), but connection between these two articles deserves further study!

[2]Both results give two explicit, combinatorial shelling orders. We give a more geometric (less explicit) shelling order. All results imply that the nested set complexes are shellable.

Acknowledgements

We thank Eva-Marie Feichtner, Caroline Klivans, and Vic Reiner for helpful conversations.

2 Illustration of Main Result

The main result uses poset combinatorics, polyhedral geometry, and some combinatorial topology. Here our goal is to illustrate the main result on a concrete example. In some parts of the example, we will use terms that have not yet been defined, but are needed to make the geometry precise. We refer the reader to Section 3 for details on all undefined notation.

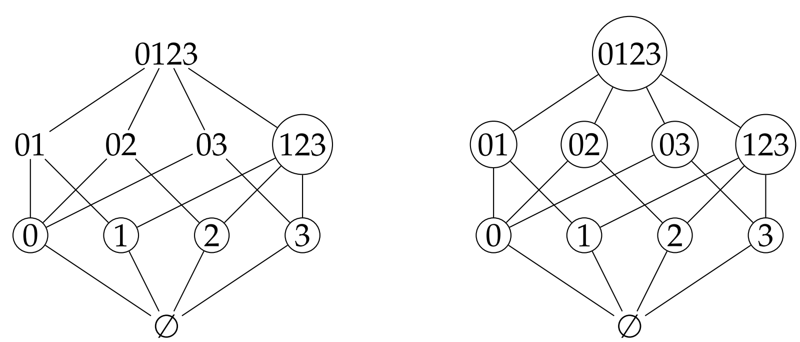
We will illustrate the main result for two different building sets of the Broom Matroid M^{br}, which is a matroid on the ground set $E = \{0, 1, 2, 3\}$ with flats

$$\emptyset, 0, 1, 2, 3, 01, 02, 03, 123, E\,.$$

We will illustrate our main theorem for the minimal building set and the maximal building set of this matroid. These building sets are

$$B^{\min} = \{0, 1, 2, 3, 123\}\,,$$
$$B^{\max} = \{0, 1, 2, 3, 01, 02, 03, 123, 0123\}\,.$$

The lattice of flats is shown below, with the two building sets circled ($B^{\min}$ is on the left and $B^{\max}$ is on the right).



2.1 Minimal Building Set

Let M^{br} be the broom matroid and $B^{\min} = \{0, 1, 2, 3, 123\}$ the minimal building set of the broom matroid. The three inclusion-maximal nested sets are $\{0, x, 123\}$ where $x = 1, 2, 3$.

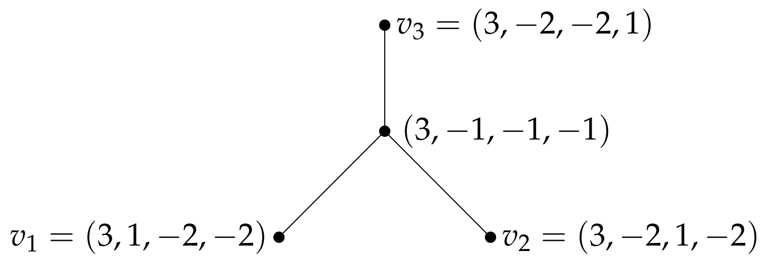
There are two maximal elements of $B^{\min}$: 0 and 123. Thus the Bergman fan of $(M^{\mathrm{br}}, B^{\min})$ has lineality space $L_{B^{\min}} = \mathrm{span}_{\mathbb{R}}\{e_0, e_1 + e_2 + e_3\}$ and three maximal cones

$$\sigma_1 = L_{B^{\min}} + \mathrm{cone}(e_1)\,, \qquad \sigma_2 = L_{B^{\min}} + \mathrm{cone}(e_2)\,, \qquad \sigma_3 = L_{B^{\min}} + \mathrm{cone}(e_3)\,.$$

Because we have a two dimensional lineality space, our normal complex will live inside a two dimensional affine linear subspace of $\mathbb{R}^4$. In this example, our normal complex will live in the the 2-dimensional affine space where $x_0 = 3$ and $x_0 + x_1 + x_2 + x_3 = 0$. In order to construct the normal complex, we take the piecewise linear function φ whose values on each cone of the Bergman fan are

$$\begin{aligned} \varphi(x) &= 3e_0^* + e_1^* - 4e_2^* && \text{for } x \in \sigma_1\,,\\ \varphi(x) &= 3e_0^* + e_2^* - 4e_1^* && \text{for } x \in \sigma_2\,,\\ \varphi(x) &= 3e_0^* + e_3^* - 4e_1^* && \text{for } x \in \sigma_3\,. \end{aligned}$$

The vertex v_i of the normal complex corresponding to the inclusion maximal nested set $\{0, i, 123\}$ is the unique point in the interior of σ_i such that $(v_i)_0 = 3$, $(v_i)_i = \varphi(e_i) = 1$ and $(v_i)_0 + (v_i)_1 + (v_i)_2 + (v_i)_3 = 0$. Inside of $\mathbb{R}^4$, the normal complex is



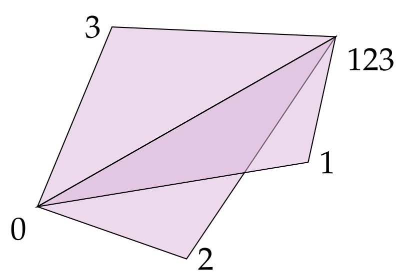
Let $\alpha = (1000, 100, 10, 1)$. Taking the inner product of α with each of the three vertices corresponding to maximal nested sets gives

$$\langle \alpha, v_1 \rangle = 3078 \qquad \langle \alpha, v_2 \rangle = 2808 \qquad \langle \alpha, v_3 \rangle = 2781$$

This induces the following order on the maximal nested sets

$$\{0, 1, 123\} < \{0, 2, 123\} < \{0, 3, 123\}\,,$$

which is a shelling order of the nested set complex (shown below).



2.2 Maximal Building Set

Let M^{br} be the broom matroid and $B^{\max}$ the maximal building set. The nested sets are the chains of the lattice of flats not containing the empty set removed. The maximal nested sets are

$$\{0,01,0123\},\{0,02,0123\},\{0,03,0123\},\{1,01,0123\},\{2,02,0123\},$$
$$\{3,03,0123\},\{1,123,0123\},\{2,123,0123\},\{3,123,0123\}\,.$$

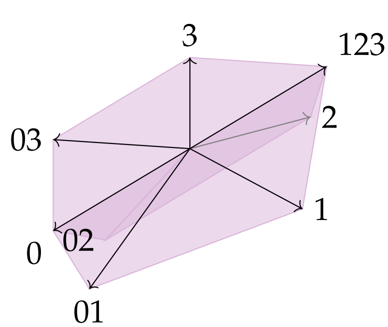
This building set has a unique maximal element, so the lineality space of the Bergman fan is the span of a single vector. That is

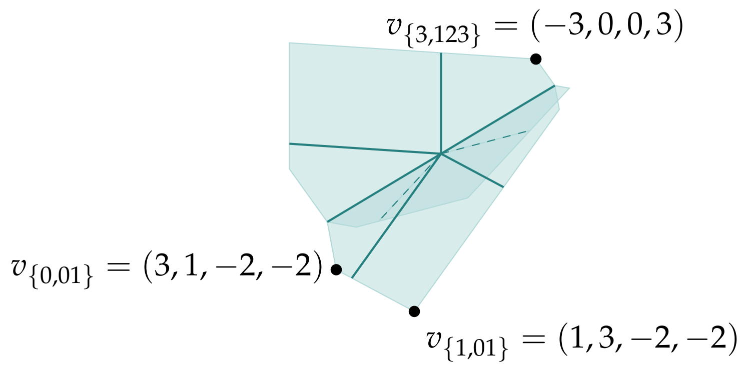
$$L_{B^{\max}} = \operatorname{span}_{\mathbb{R}}(e_0 + e_1 + e_2 + e_3)\,.$$

The fan itself has 9 maximal cones, corresponding to the 9 maximal nested sets listed above. Below, on the left, we draw a truncated piece of the Bergman fan (where the rays are labeled by the flats they correspond to). Let φ be the piecewise linear function defined on maximal cones of this Bergman fan by

$$\begin{aligned}
\varphi(x) &= 3e_0^* + e_1^* - 4e_2^* && \text{for } x \in \sigma_{0,01}\,,\\
\varphi(x) &= 3e_0^* + e_2^* - 4e_1^* && \text{for } x \in \sigma_{0,02}\,,\\
\varphi(x) &= 3e_0^* + e_3^* - 4e_1^* && \text{for } x \in \sigma_{0,03}\,.\\
\varphi(x) &= 3e_1^* + e_0^* - 4e_2^* && \text{for } x \in \sigma_{1,01}\,,\\
\varphi(x) &= 3e_2^* + e_0^* - 4e_1^* && \text{for } x \in \sigma_{2,02}\,,\\
\varphi(x) &= 3e_3^* + e_0^* - 4e_1^* && \text{for } x \in \sigma_{3,03}\,,\\
\varphi(x) &= 3e_1^* - 3e_0^* && \text{for } x \in \sigma_{1,123}\,,\\
\varphi(x) &= 3e_2^* - 3e_0^* && \text{for } x \in \sigma_{2,123}\,,\\
\varphi(x) &= 3e_3^* - 3e_0^* && \text{for } x \in \sigma_{3,123}\,.
\end{aligned}$$

On the right, we pick a cubical function and show the normal complex with respect to that cubical function.





Let $\alpha = (1000, 100, 10, 1)$. Taking the inner product of α with each of the three vertices corresponding to maximal nested sets gives

$$\begin{aligned}
\langle \alpha, v_{\{0,01,0123\}} \rangle &= 3078 & \langle \alpha, v_{\{0,02,0123\}} \rangle &= 2808 \\
\langle \alpha, v_{\{0,03,0123\}} \rangle &= 2781 & \langle \alpha, v_{\{1,01,0123\}} \rangle &= 1278 \\
\langle \alpha, v_{\{2,02,0123\}} \rangle &= 828 & \langle \alpha, v_{\{3,03,0123\}} \rangle &= 783 \\
\langle \alpha, v_{\{1,123,0123\}} \rangle &= -2700 & \langle \alpha, v_{\{2,123,0123\}} \rangle &= -2970 \\
\langle \alpha, v_{\{3,123,0123\}} \rangle &= -2997
\end{aligned}$$

This induces the following order on the maximal nested sets

$$\begin{aligned}
\{0, 01, 0123\} <&\{0, 02, 0123\} < \{0, 03, 0123\} < \{1, 01, 0123\} < \{2, 02, 0123\} \\
&<\{3, 03, 0123\} < \{1, 123, 0123\} < \{2, 123, 0123\} < \{3, 123, 0123\} \,,
\end{aligned}$$

and we can check that this is a shelling order of the nested set complex.

Since we have the maximal nested set complex, this simplicial complex is a cone over the order complex of the proper part of the lattice of flats. Famously, Björner gives a shelling order on the order complex on the lattice of flats with respect to any linear order on the ground set. Taking $0 \prec 1 \prec 2 \prec 3$ as our order, Björner's shelling order is

$$\begin{aligned}
\{0, 01, 0123\} <&\{0, 02, 0123\} < \{0, 03, 0123\} < \{1, 01, 0123\} < \{1, 123, 0123\} \\
&<\{2, 02, 0123 < \{2, 123, 0123\}\} < \{3, 03, 0123\} < \{3, 123, 0123\} \,.
\end{aligned}$$

This is a different order than our shelling order coming from the normal complex, even though they have the same first element.

Note that the lexicographic (Definition 6) condition is necessary. If $\alpha = (1, 100, 101, -1000)$ (this vector is not lexicographic), then the first two maximal nested sets in the induced order are $\{2, 02, 0123\} < \{1, 01, 0123\}$. We can already see that the induced order fails to be a shelling order because the first two maximal nested sets don't intersect!

3 Normal Complexes

The goal of this section is to describe the normal complex of the pair $(\mathcal{L}, B)$. This is a polytopal complex that plays the role of a normal polytope for complete fans (our fans are sometimes not complete, and so may not be the normal fan of a polytope). The full construction of normal complexes is too technical for this short abstract, but is a distilled version of the one given by the third author and Ross in [19]. The important information for our context is encapsulated in Theorem 4, below.

In short, this complex will be defined by a particular type of function ℓ mapping the Bergman fan to $\mathbb{R}$. Such a function is called ***piecewise linear*** if for every cone σ, the restriction $\ell|_\sigma$ is a linear function on $\operatorname{span}(\sigma)$. Denote the space of piecewise linear functions on a fan Σ as $\mathrm{PL}(\Sigma)$.

In order to state Theorem 4, we denote the ***restriction*** of the lattice of flats $\mathcal{L}$ to some flat X with $\mathcal{L}|_X = \{Z \in \mathcal{L} \colon Z \leq X\}$ and the ***contraction*** as $\mathcal{L}^X = \{Z \in \mathcal{L} \colon X \vee Z \in \mathcal{L}\}$. We can restrict and contract building sets and nested sets to an element X as long as it is a member of the respective set. Note that lattices of flats, building sets, and nested sets remain such under these operations [14, 8], and hence, we can talk about the nested set complexes $\Delta(\mathcal{L}|_X, B|_X)$ and $\Delta(\mathcal{L}^X, B^X)$.

Theorem 4 ([19, Section 2]). *For every pair $(\mathcal{L}, B)$, there exists a function $\varphi \in \mathrm{PL}(\Sigma_{\mathcal{L},B})$ that defines a collection of halfspaces H_X^+, one for each flat $X \in B \setminus \max B$, and an affine subspace H_L orthogonal to the lineality space L of the Bergman fan $\Sigma_{\mathcal{L},B}$. The* **normal complex**

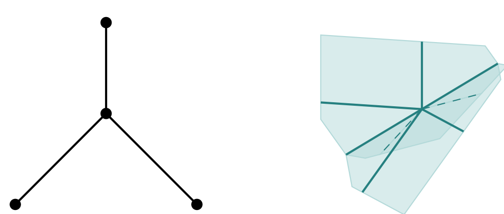
$$\mathcal{N}_{\mathcal{L},B,\varphi} := \Sigma_{\mathcal{L},B} \cap \bigcap_{X \in B \setminus \max(B)} H_X^+ \cap H_L$$

is a polytopal complex satisfying the following properties:

- *For every maximal cone σ_N of $\Sigma_{\mathcal{L},B}$, the complex $\mathcal{N}_{\mathcal{L},B,\varphi}$ has an extreme vertex v_N which sits in the relative interior of σ.*
- *The intersection of $\mathcal{N}_{\mathcal{L},B,\varphi}$ with any of the hyperplanes H_X defining some H_X^+ is the normal complex of the direct product of the restriction $(\mathcal{L}|_X, B|_X)$ to X with the contraction $(\mathcal{L}/X, B/X)$ at X.*

When such a φ defines a normal complex, we call it a **cubical function**.

Example 5. In Section 2, we looked at two different building sets on the broom matroid. We found cubical functions for each of these building sets and constructed the normal complexes. The normal complexes are shown below with the $B^{\min}$ normal complex on the left and the $B^{\max}$ normal complex on the right.



We are now ready to describe the order on the facets of the nested set complexes, which will turn out to be a shelling order. This order comes from taking inner-products of the distinguished vertices of the normal complex with a special vector.

Definition 6. Let $\alpha \in \mathbb{R}^E$ and $\mathcal{N}_{\mathcal{L},B,\varphi}$ be a normal complex. We say that α is ***lexicographic on*** $\mathcal{N}_{\mathcal{L},B,\varphi}$ if for every pair of facets N, N' of $\Delta(\mathcal{L}, B)$ and the corresponding pair of vertices $v_N, v_{N'}$ of $\mathcal{N}_{\mathcal{L},B,\varphi}$, we have

$$\langle \alpha, v_N \rangle > \langle \alpha, v_{N'} \rangle$$

if and only if there exists an index k such that $(v_N)_k > (v_{N'})_k$ and for all $1 \leq i < k$, we have $(v_N)_i = (v_{N'})_i$.

Note 7. In our proof of the main theorem, we refer to the fact that lexicographic vectors “restrict well" with respect to the normal complex. That is, if α is lexicographic on $\mathcal{N}_{\mathcal{L},B,\varphi}$ then, for any supporting hyperplane H_X, it is lexicographic on the face $\mathcal{N}_{\mathcal{L},B,\varphi} \cap H_X$.

Formally, if X is a proper and non-empty flat in $\mathcal{L}$, then α restricts to a lexicographic vector on the link of X in $\mathcal{N}$, i.e., the restriction $\alpha|_X := \pi_F(\alpha) \in \mathbb{R}^F$ and contraction $\alpha/X := \pi_{E\setminus X}(\alpha) \in \mathbb{R}^{E\setminus F}$ combine to give a lexicographic vector on $\mathcal{N}_{\mathcal{L}|_X,B|_X,\varphi|_X} \times \mathcal{N}_{\mathcal{L}/X,B/X,\varphi/X}$. See Lemma 9 to see that the product of normal complexes is a product.

4 An Order on the Facets of the Nested Set Complex

In this section, we define a linear order on the facets of the nested set complex. Our facet order is inspired by work of Bruggesser–Mani in [9]; see also [21, Theorem 8.11]. Namely, we start with a particular type of vector α and order the facets of $\Delta(\mathcal{L}, B)$ by taking inner products of α this the vertices of the normal complex. This induces a linear order on the facets of $\Delta(\mathcal{L}, B)$. In the proof of Theorem 1, we will show that this order is a shelling order for the nested set complex.

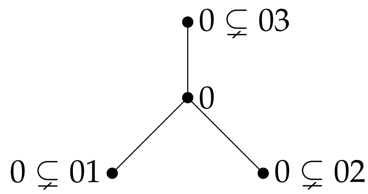
In order to define a linear order $<_\alpha$ on the distinguished vertices of $\mathcal{N}_{\mathcal{L},B,\varphi}$ with respect to a lexicographic vector α, say that

$$F <_\alpha G \qquad \Longleftrightarrow \qquad \langle v_{F,B,\varphi}, \alpha \rangle > \langle v_{G,B,\varphi}, \alpha \rangle \,.$$

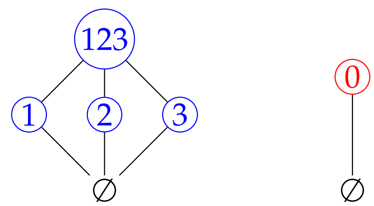
We call this the ***normal complex order*** (with respect to α).

In the proof of Theorem 1, we will show that the normal complex order is a shelling order for any lexicographic vector. The proof is somewhat subtle and depends strongly on the geometry of normal complexes, but allows us to use combinatorics in the inductive argument. The key observation is that these restrictions are honest restrictions to smaller normal complexes. We illustrated this first with an example and then state the key technical result that guarantees our approach works, Lemma 9.

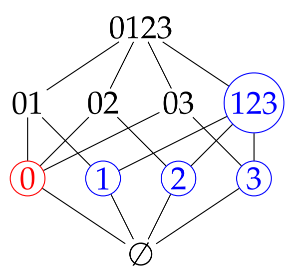
Example 8. Let us return to the broom matroid with maximal building set introduced above. Take M to be the broom matroid, φ a cubical function for M with respect to the maximal building set $B^{\max}$ and $\mathcal{N}_{M,B^{\max},\varphi}$ the resulting normal complex. Below we draw the intersection of $\mathcal{N}_{M,B^{\max},\varphi}$ with the hyperplane defined by $x_0 = \varphi(0)$ and label the vertices by their corresponding chains.



This is the direct product of two normal complexes: the normal complex of the deletion at 0 and the direct product of the contraction at 0, both with respect to the maximal building set. Below, we draw the corresponding lattices of flats and with their building sets circled.



This direct product of normal complexes corresponds to a normal complex of the product of the two posets with the building set $B = \{0, 1, 2, 3, 123\}$, the elements of which are circled in the picture below.



Note that this building set is precisely the first building set from Section 2.

In this example, we use the fact that direct product of normal complexes is again a normal complex. This holds in general, as we explain below.

Lemma 9. *The direct product of normal complexes is a normal complex.*

This lemma is partly geometric and partly combinatorial. On the combinatorics side, we use the fact that nested set complexes behave well under taking direct products. The products of nested set complexes used in the previous example was studied by de Concini–Procesi [12] in the realizable case, then generalized to arbitrary matroids by Bibby–Denham–Feichtner [4, Propositions 2.8.6-7] and their work was reinterpreted by Brauner–Eur–Pratt–Vlad in [8, Theorem 1.6] via links of flats in nested set complexes. These constructions are also an important part of work by Mantovani–Pardol–Pilaud on *acyclonestohedra* [17, Proposition 4.18].

By choosing a lexicographic weight vector to sweep through the normal complex, for example, it turns out that we fix the first element of shelling order. This first element turns out to have an easy (but slightly involved to state) combinatorial description. This provides another concrete way to relate the polyhedral geometry of normal complexes to the combinatorics and topology of nested set complexes. That is, we obtain the following useful lemma.

Lemma 10. *The first facet of the normal complex order has a purely combinatorial description (we omit the description from this abstract, but note that it is something concrete that one can describe explicitly in terms of poset combinatorics).*

5 Proof Idea for Theorem 1

The goal of this section is to not to prove Theorem 1, but rather to explain the idea of the proof and present several of the key technical lemmas.

The proof of Theorem 1 comes from an induction argument on the dimension of the normal complex (where dimension is taken after quotienting out by the lineality space). This induction on dimension is subtly different than induction on the rank of the matroid, since it depends not only on the rank of the matroid but also on the number of maximal elements in the building set. As we saw in Section 2, for example, two different building sets of the same matroid may have normal complexes of different dimension.

The base case of the induction is relatively straightforward and comes from the following lemma.

Lemma 11. *Let $\mathcal{N} = \mathcal{N}_{\mathcal{L},B,\varphi}$ be a normal complex of dimension 1. Then the maximal nested sets have the form $\max(B) \cup \{F\}$ for $F \in B \setminus \max(B)$. Furthermore, if $F \in B \setminus \max(B)$ then F is an atom of M.*

Said differently, Lemma 11 says that if $\mathcal{N}$ is a normal complex of dimension 1 then any order on the facets of the associated nested set complex Δ will be a shelling order of the simplicial complex Δ.

For the inductive step, we make heavy use of the restrictions to smaller normal complexes and the results from Theorem 4, Lemma 9, and Lemma 10. The main technical claim that goes into Theorem 1 says that sweeping a lexicographic vector through a normal complex gives a shelling order on the nested set complex. More precisely, we have the following claim.

Claim 12. *Let $\mathcal{N} = \mathcal{N}_{\mathcal{L},B,\varphi}$ be the normal complex of a matroid M with respect to a building set B and cubical function $\varphi \in PL(\Sigma_{B,M})$. Fix a lexicographic vector α on $\mathcal{N}$. The normal complex order induced by α is a shelling order of the nested set complex.*

From here, the proof of Theorem 1 is immediate. Any choice of such α gives a shelling order of the nested set complex, so nested set complexes are shellable.

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