Real Powers of Monomial Ideals

by Josiah Lim, Ethan Roy and Ethan Partida
on Mar 14, 2021

» Outline

* Background
* Geometric Representations
* Real Powers
* Jumping numbers

Background

- What is a Monomial?
- What is a Monomial Ideal?

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* What is a Monomial Ideal?

» What is a monomial?

Definition (Monomial)

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Examples

x^4, x^2y and xyz are monomials.

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Examples

x^4, x^2y and xyz are monomials.

Non-examples

$x+y$ and $xy-z$ are polynomials, not monomials.

Background

* What is a Monomial?
* What is a Monomial Ideal?

» What is a Monomial Ideal?

Definition (Monomial Ideal *Spooky*)

Let $M = \{\mathbb{m}_1, \ldots, \mathbb{m}_k\}$ be a set of monomials. The ideal generated by M, written $I = (\mathbb{m}_1, \ldots, \mathbb{m}_k)$, is the set containing all polynomials which have the form $p_1\mathbb{m}_1 + \ldots + p_k\mathbb{m}_k$ where each p_i is a polynomial.

Geometric Representations

* Monomials and Monomial Ideals $\longrightarrow$ Staircases
* What is a Newton Polytope?
* What is a Newton Polyhedron?

Geometric Representations

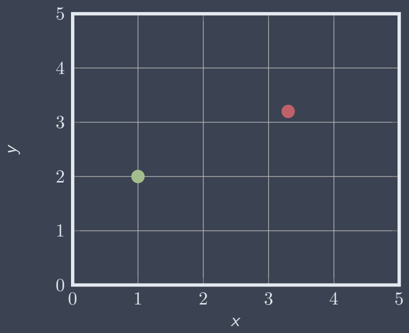
- ∗ Monomials and Monomial Ideals $\longrightarrow$ Staircases
- ∗ What is a Newton Polytope?
- ∗ What is a Newton Polyhedron?

» Lattice Points

Definition (Lattice Point)

A point $x \in \mathbb{R}^n$ is a lattice point if it has integer coordinates

Green: Lattice Point, Red: Not a Lattice Point



» Monomials ⟶ Lattice Points

* Each monomial has a corresponding lattice point.
* A monomial $x^a y^b$ corresponds to (a, b)

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Example

In the xy plane,

$$xy^3 \to (1,3)$$
$$x^2y \to (2,1)$$
$$x^4 \to (4,0)$$

» Generators ⟶ Lattice Points

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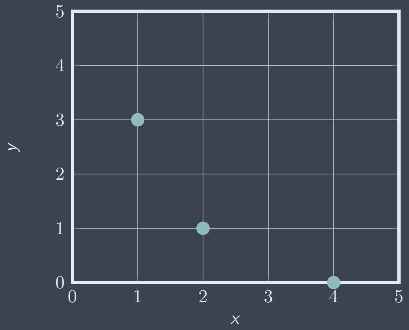
» Generators ⟶ Lattice Points

* Monomials ideals may seem complicated, but pictures are not!
* For the ideal $I = (xy^3, x^2y, x^4)$, the generators are $(1, 3)$, $(2, 1)$ and $(4, 0)$.

Generators of the ideal $I = (xy^3, x^2y, x^4)$



» Divisibility

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* $x^2y^2 = y \cdot x^2y \in (xy^3, x^2y, x^4)$
* $x^3y = x \cdot x^2y \in (xy^3, x^2y, x^4)$

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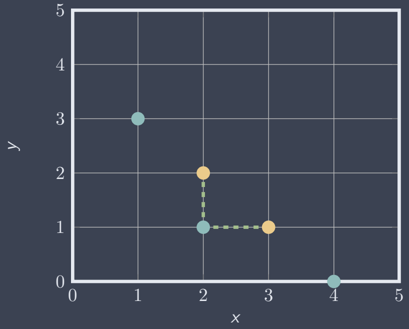
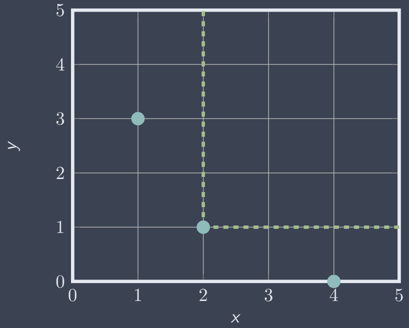


» Divisibility

* Moving up and to the right corresponds to multiplying by x and y respectively
* Monomials divisible by a generator are those up and to the right of it

Monomials divisible by x^2y



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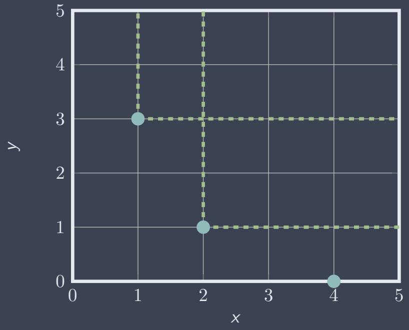
Monomials divisible by x^2y or xy^3



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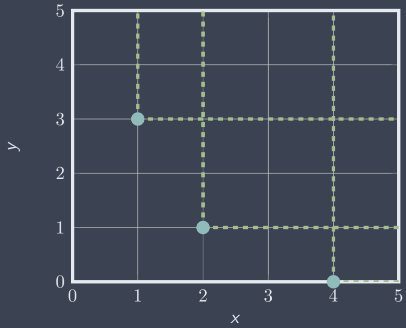
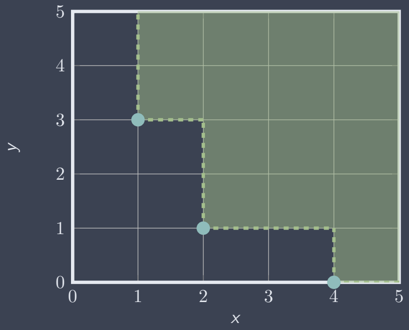


» Ideals ⟶ Staircases

Filling in these boxes, we get a monomial ideal's corresponding staircase[1].

The staircase of (x^2y, xy^3, x^4)

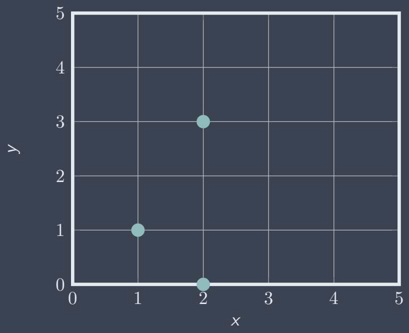


[1]See [Mil05, Chapter 3] for lots of interesting properties of these staircases

» Points ⟶ Ideals

Given a set of points, we can construct a monomial ideal by looking at the staircase the points generate.

The points $(1, 1), (2, 0), (2, 3)$



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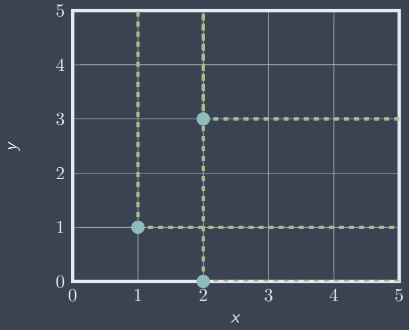
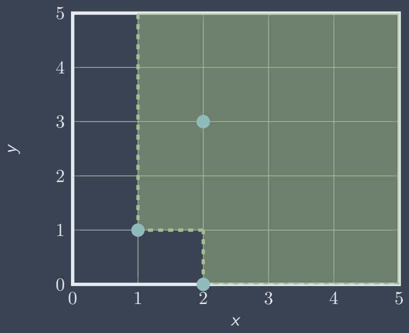
The staircases of $(1,1),(2,0),(2,3)$



» Points ⟶ Ideals

Given a set of points, we can construct a monomial ideal by looking at the staircase the points generate.

The staircase of $(xy, x^2, x^2y^3) = (xy, x^2)$



Geometric Representations

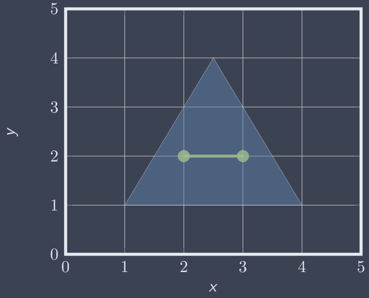
* Monomials and Monomial Ideals $\longrightarrow$ Staircases
* What is a Newton Polytope?
* What is a Newton Polyhedron?

» Convexity

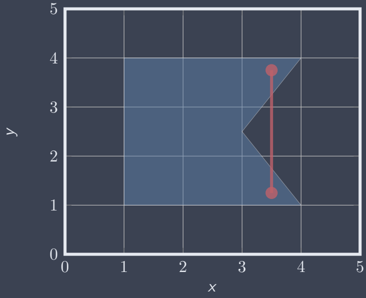
Definition (Convex)

A space $K \subseteq \mathbb{R}^n$ is convex if $\forall a, b \in K$, the line between a and b is contained in K.

Left: Convex



Right: Not Convex



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Definition (Convex Hull)

The convex hull of $K \subseteq \mathbb{R}^n$ is the smallest convex space containing K.

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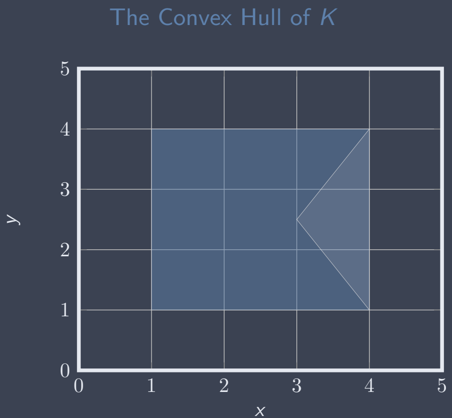
The convex hull of $K \subseteq \mathbb{R}^n$ is the smallest convex space containing K.

This is the space formed by "wrapping a rubber band around K".

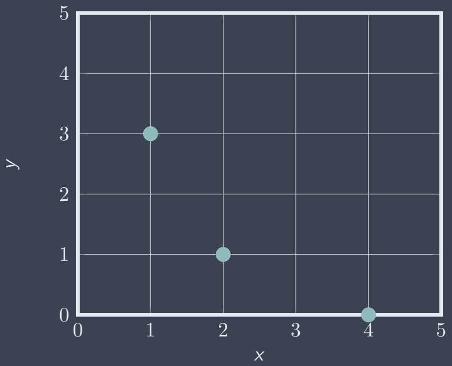


» What is a Newton Polytope?

Definition (Newton Polytope)

The Newton polytope of an ideal I, $np(I)$, is the convex hull of its generators.

Generators of (xy^3, x^2y, x^4)



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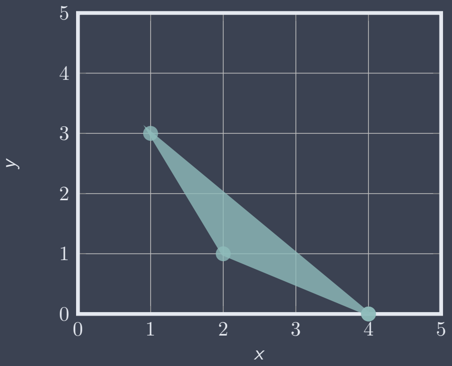

Geometric Representations

- Monomials and Monomial Ideals $\longrightarrow$ Staircases
- What is a Newton Polytope?
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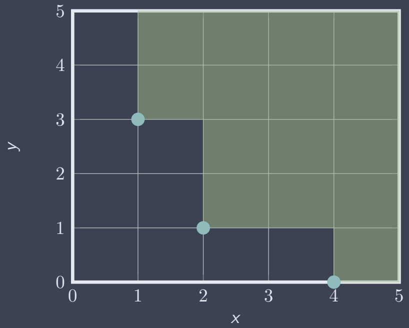
The Newton polyhedron of an ideal I, $NP(I)$, is the convex hull of its staircase.

The staircase of (x^2y, xy^3, x^4)



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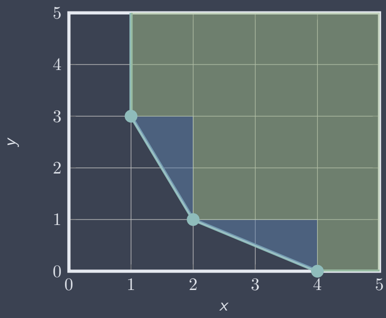

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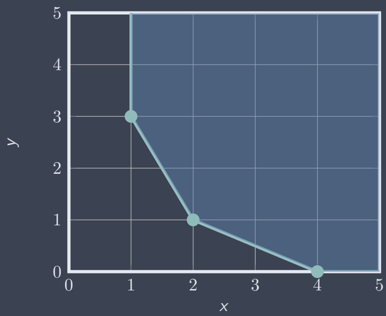
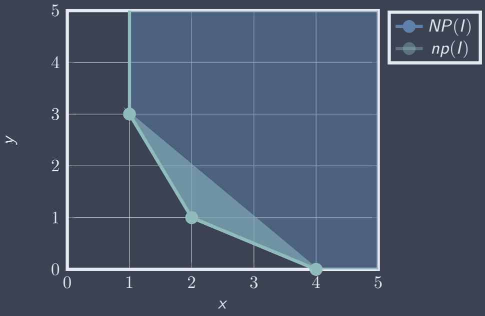


» NP vs np

$NP(I)$ can be thought of as an extension of $np(I)$, everything up and to the right.

Newton Polyhedron and Newton Polytope of $I = (xy^3, x^2y, x^4)$



Real Powers

* What is a Real Power?
* Computing Real Powers

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» What is a Real Power?

Definition (Real Power)

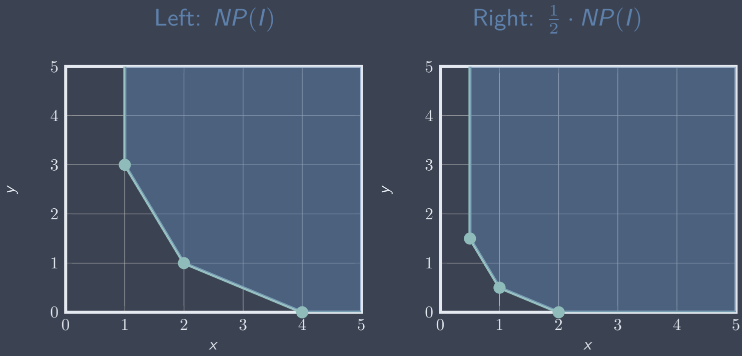
The real power r of an ideal I, $\overline{I^r}$, is the ideal corresponding to the staircase of the lattice points in $r \cdot NP(I)$.

» Example of Real Power

Let's compute $\overline{(xy^3, x^2y, x^4)^{\frac{1}{2}}}$.

» Example of Real Power

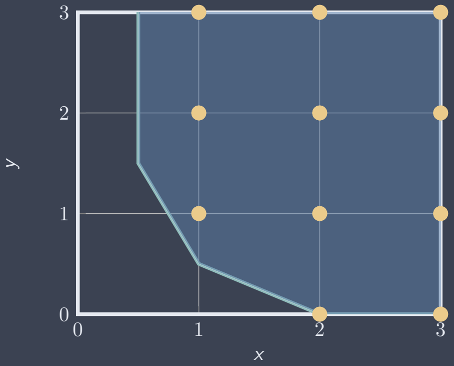
Let's compute $\overline{(xy^3, x^2y, x^4)^{\frac{1}{2}}}$. We first need to find $\frac{1}{2} \cdot NP(I)$.



» Example of Real Power

We now identify the lattice points and draw their staircases.

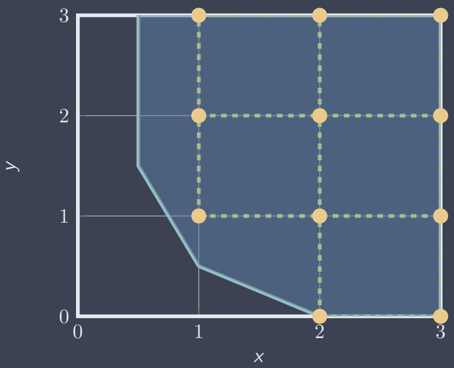
Lattice points of $\frac{1}{2} \cdot NP(x^4, x^2y, xy^3)$



» Example of Real Power

We now identify the lattice points and draw their staircases.

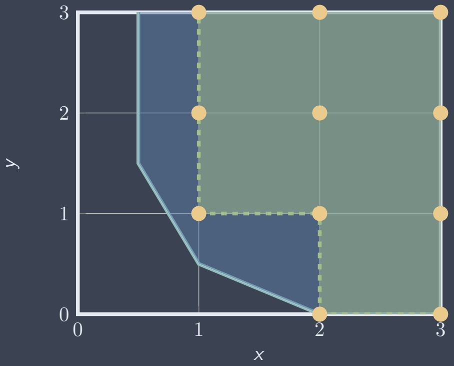
Staircases of $\frac{1}{2} \cdot NP(x^4, x^2y, xy^3)$



» Example of Real Power

We now identify the lattice points and draw their staircases.

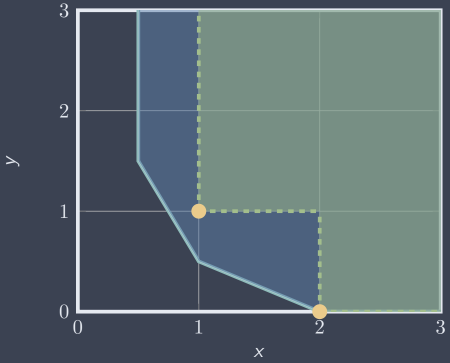
The staircase of our lattice points



» Example of Real Power

Thus $\overline{(x^4, x^2y, xy^3)^{\frac{1}{2}}} = (xy, x^2)$.

The staircase of $\overline{(x^4, x^2y, xy^3)^{\frac{1}{2}}} = (xy, x^2)$



Real Powers

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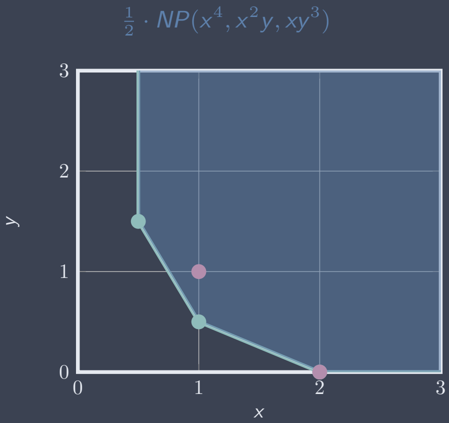
1. Not much is known about real powers
2. Looking for Patterns
3. Patterns require lots of examples
4. Examples are hard to compute
   * computer programs are FASTER than working it out by hand

» Example

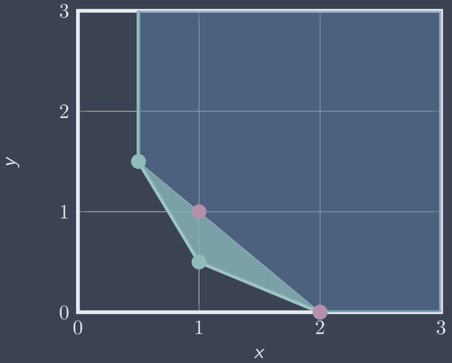
Notice that the minimal generators are "close" to the boundary of $NP(I)$.



» Example

Could the generators of $r \cdot NP(I)$ be in $r \cdot np(I)$?

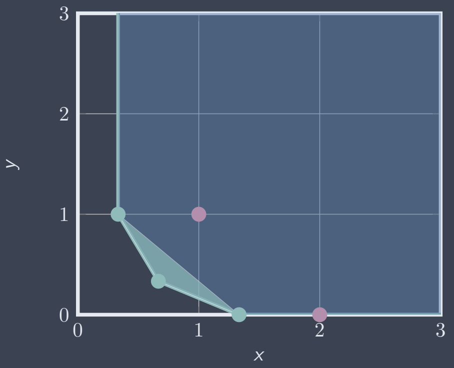
$\frac{1}{2} \cdot NP(x^4, x^2y, xy^3)$ and $\frac{1}{2} \cdot np(x^4, x^2y, xy^3)$



» Example

No:(

$\frac{1}{3} \cdot NP(x^4, x^2y, xy^3)$ and $\frac{1}{3} \cdot np(x^4, x^2y, xy^3)$



» Bounding Theorem

But very close!

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Bounding Theorem (Real Powers Team [Don+21])

Let $r = \frac{p}{q} \in \mathbb{Q}$ and I be a monomial ideal in d variables.
Then the minimal generators of $\overline{I^r}$ are within a distance of $(d - \frac{1}{q})$ from $r \cdot np(I)$.

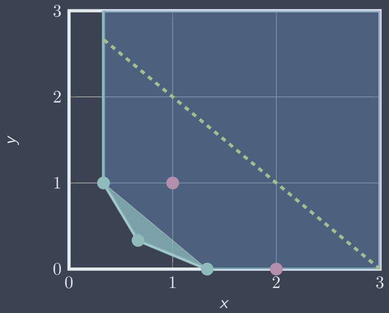
» Bounding Theorem

For $I = (x^4, x^2y, xy^3)$ and $r = \frac{1}{3}$, minimal generators will be within a distance of $2 - \frac{1}{3}$ from $r \cdot np(I)$.

» Bounding Theorem

For $I = (x^4, x^2y, xy^3)$ and $r = \frac{1}{3}$, minimal generators will be within a distance of $2 - \frac{1}{3}$ from $r \cdot np(I)$.

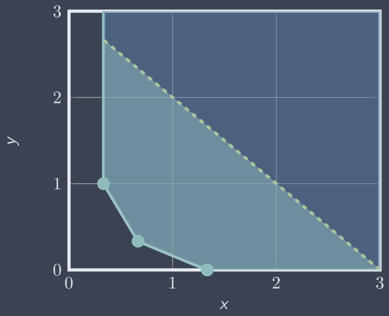
Minimal generators of $r \cdot NP(I)$



» Algorithm Steps

1. Find all lattice points within the bounded distance of $r \cdot np(I)$ (Minkowski Sum)

The bounded distance from $r \cdot np(I)$



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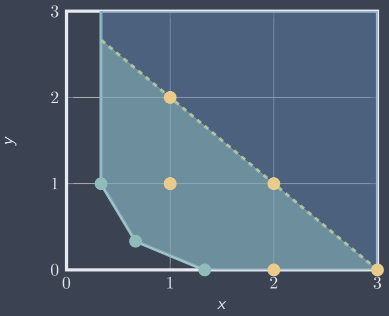
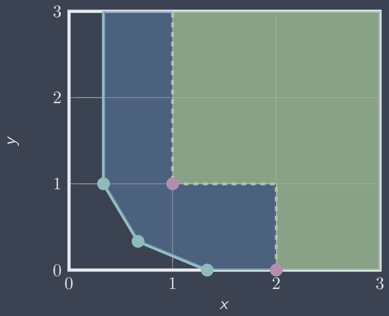
The lattice points within a bounded distance of $r \cdot np(I)$



» Algorithm Steps

1. Find all lattice points within the bounded distance of $r \cdot np(I)$ (Minkowski Sum)
2. Compute the ideal corresponding to the staircase of these points

The staircase of $\overline{(x^4, x^2y, xy^3)^{\frac{1}{3}}} = (x^2, xy)$



Jumping Numbers

* What is a Jumping Number?
* A reformulation
* Corollaries
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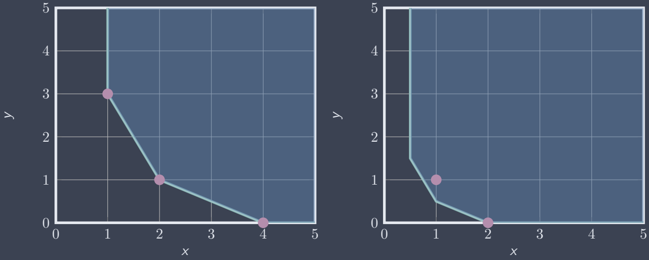
» Jumping Numbers

Left: $NP(I)$ Right: $\frac{1}{2} \cdot NP(I)$



» What is a Jumping Number?

Definition (Jumping Number)

We say that a number r is a jumping number if $\overline{I^r} \neq \overline{I^{r+\epsilon}}$ for all $\epsilon > 0$.

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We say that a number r is a jumping number if $\overline{I^r} \neq \overline{I^{r+\epsilon}}$ for all $\epsilon > 0$.

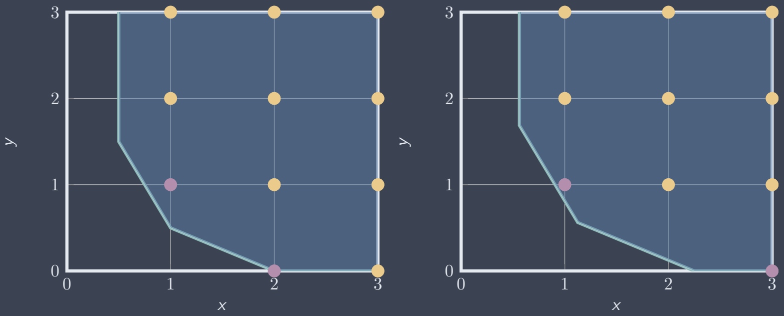
For $I = (x^4, x^2y, xy^3)$, we have that

* $\frac{1}{2}$ is a jumping number
* $\frac{1}{3}$ is not a jumping number

» $\frac{1}{2}$ is a Jumping Number

Increasing $\frac{1}{2}$ just a little bit will no longer include the point $(2,0)$. This removes a minimal generator and changes the ideal.

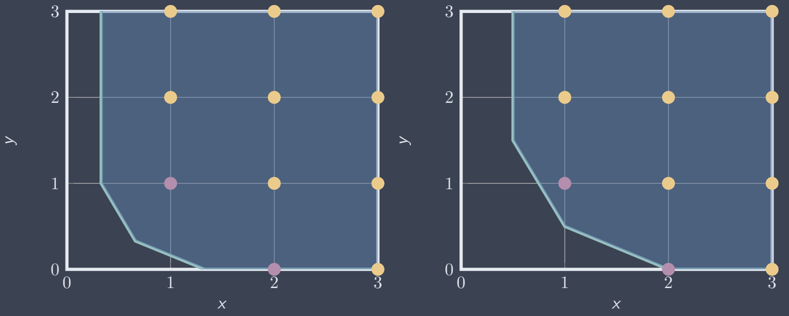
$r = \frac{1}{2}$ vs $r = \frac{5}{8}$



» $\frac{1}{3}$ is not a Jumping Number

- By looking at $r \cdot NP(I)$ we can determine $\overline{I^r}$.
- We can see, $\overline{I^{\frac{1}{2}}} = \overline{I^{\frac{1}{3}}}$.

$r = \frac{1}{3}$ vs $r = \frac{1}{2}$



Jumping Numbers

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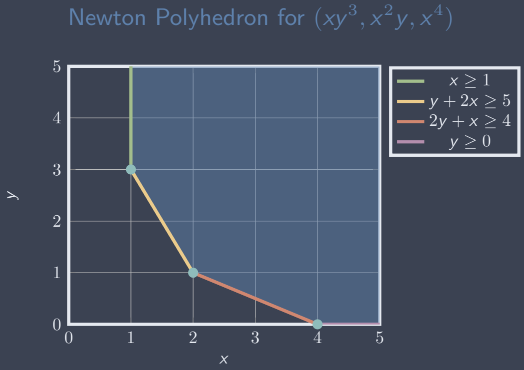
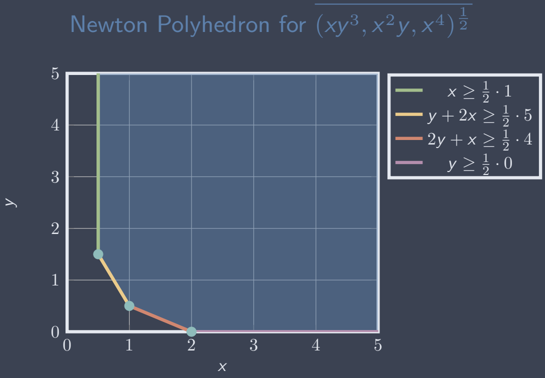
» A new perspective

We can describe Newton polyhedron by a system of linear inequalities.



» A new perspective

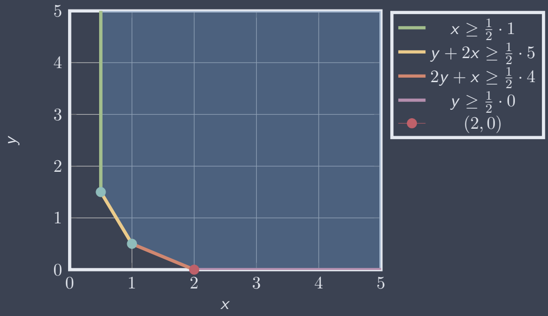
Scaling a Newton polyhedron corresponds to scaling constants in our inequalities.



» A new perspective

Notice that $(2, 0)$ gives equality in $2y + x \geq \frac{1}{2} \cdot 4$ and inequality for the rest of our bounding equations.

Newton Polyhedron for $\overline{(xy^3, x^2y, x^4)^{\frac{1}{2}}}$



» A new perspective

Theorem (Real Powers Team [Don+21])

$r \in \mathbb{R}$ is a jumping number for a monomial ideal I if and only if there is an integer solution to the bounding inequalities of $r \cdot NP(I)$ which gives equality to one of the (interesting) inequalities.

» Example

With $I = (x^4, x^2y, xy^3)$, the system of linear inequalities for $\overline{I^{\frac{1}{2}}}$ is:

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With $I = (x^4, x^2y, xy^3)$, the system of linear inequalities for $\overline{I^{\frac{1}{2}}}$ is:

$$x \geq \frac{1}{2}$$
$$y + 2x \geq \frac{5}{2}$$
$$2y + x \geq 2$$
$$y \geq 0$$

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$$x \geq \frac{1}{2}$$
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$$2y + x \geq 2$$
$$y \geq 0$$

Notice $x = 2, y = 0$ is a solution to these inequalities that gives equality in $2y + x = 2$.

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$$
\begin{aligned}
x &\geq \frac{1}{3} \\
y + 2x &\geq \frac{5}{3} \\
2y + x &\geq \frac{4}{3} \\
y &\geq 0
\end{aligned}
$$

There are no integer solutions since no product and sum of integers is a non-integer.

Jumping Numbers

* What is a Jumping Number?
* A reformulation
* Corollaries
* Worked Example

» Rationality

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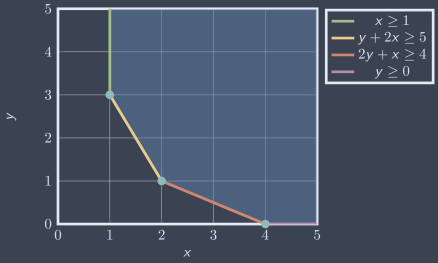
* All jumping numbers are rational.
* For each $r \in \mathbb{R}_+$ there exists $r' \in \mathbb{Q}$ so that $\overline{I^r} = \overline{I^{r'}}$.

Newton Polyhedron for (xy^3, x^2y, x^4)



» Linearity

Since $NP(I)$ is a linear system of inequalities, a notion of multiplicity exists for jumping numbers.

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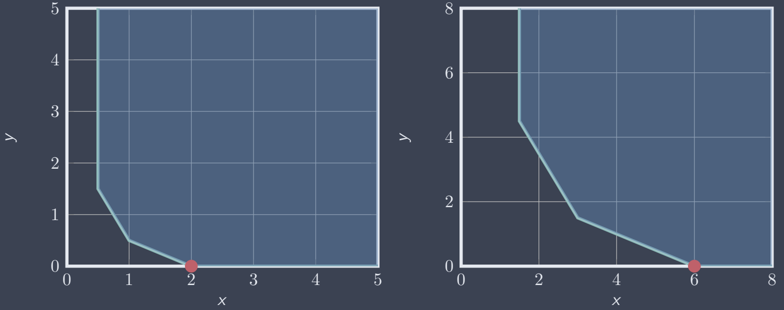
Linearity Result (Real Powers Team [Don+21])

If r is a jumping number of I then nr is also a jumping number of I for all $n \in \mathbb{N}$.

$\frac{1}{2} \cdot NP(xy^3, x^2y, x^4)$ and $\frac{3}{2} \cdot NP(xy^3, x^2y, x^4)$



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For example, $2y + x = \frac{8}{5}$ has no integer solutions.

» Example

Using these results, we can systematically determine all jumping numbers of a given ideal. We use this to find all jumping numbers of $I = (x^4, x^2y, xy^3)$ that are less than one.

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$$x \geq 1$$
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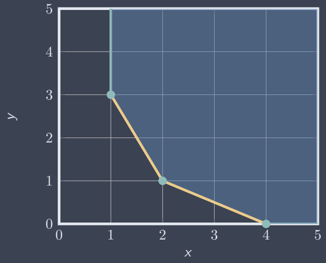
$$2y + x \geq 4$$
$$y \geq 0$$



» Scaled Inequalities

Scaling our Newton polyhedron by r we get the following system of inequalities:

$$x \geq r \cdot 1$$
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The restriction lemma limits our jumping numbers to those of the form: $\frac{n}{5}$ or $\frac{m}{4}$ for $n, m \in \mathbb{N}$.

» Possible Jumping Numbers

So our possible jumping numbers less than 1 are:

$$\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}.$$

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We can now manually check the validity of each of these possible jumping numbers.

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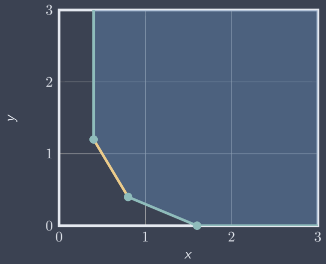

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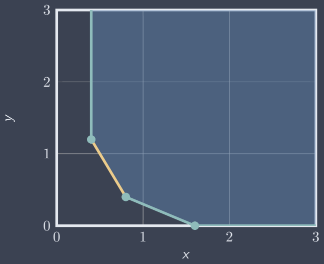

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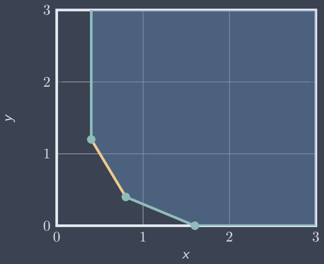

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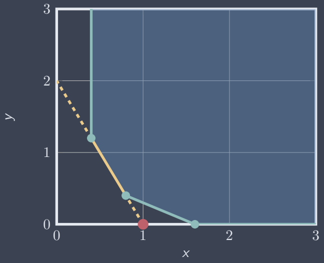

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$$\{\frac{n}{4}, \frac{m}{5} : n \geq 1, m \geq 3\}.$$

Conclusion

» Future Work

* Jumping Numbers Algorithm

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* Numerical Semi-Groups

» Future Work

- Jumping Numbers Algorithm
- Numerical Semi-Groups
- Jumping Numbers for classes of ideals (pure, squarefree, etc.)

» References

Sturmfels Miller. *Combinatorial Commutative Algebra*. Springer, 2005.

Pratik Dongre et al. *Real powers of monomial ideals*. 2021. arXiv: 2101.10462 [math.AC].

» Thank You!

Any Questions?