Rational Powers of Monomial Ideals

Josiah Lim, Ethan Partida, Ethan Roy
2020 Polymath REU

Background

Monomial

A *monomial* is a product of variables.

- x^4, x^2y and x^2yz^3 are monomials.
- $x + x^2$ and $xy^2 - z$ are polynomials, not monomials.

A monomial is denoted using the shorthand (vector) notation

$$\mathbf{x}^{\mathbf{a}} = {x_1}^{a_1}{x_2}^{a_2} \dots {x_k}^{a_k}.$$

With this notation, each monomial $\mathbf{x}^{\mathbf{a}}$ corresponds to the point $(a_1, \dots, a_k)$. For example, in the x-y plane,

$$x^4 \to (4, 0),$$
$$x^2y \to (2, 1).$$

Monomial Ideal

Let $M = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_n}\}$ be a set of monomials. The *monomial ideal* generated by M, written $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_n})$, contains all polynomials which have the form $p_1\mathbf{x}^{\mathbf{a}_1} + \dots + p_n\mathbf{x}^{\mathbf{a}_n}$ where each p_i is a polynomial.

Convex Representations

Monomials ideals, like monomials, can also be visualized on the coordinate plane.

- The *Newton Polytope* of an ideal, written $np(I)$, is the convex hull of the minimal generators of I.
- The *Newton Polyhedron* of an ideal, written $NP(I)$, is the convex hull of all monomials in I.

Figure 1 shows $np(I)$ and $NP(I)$ when $I = (x^4, x^2y, xy^3)$.

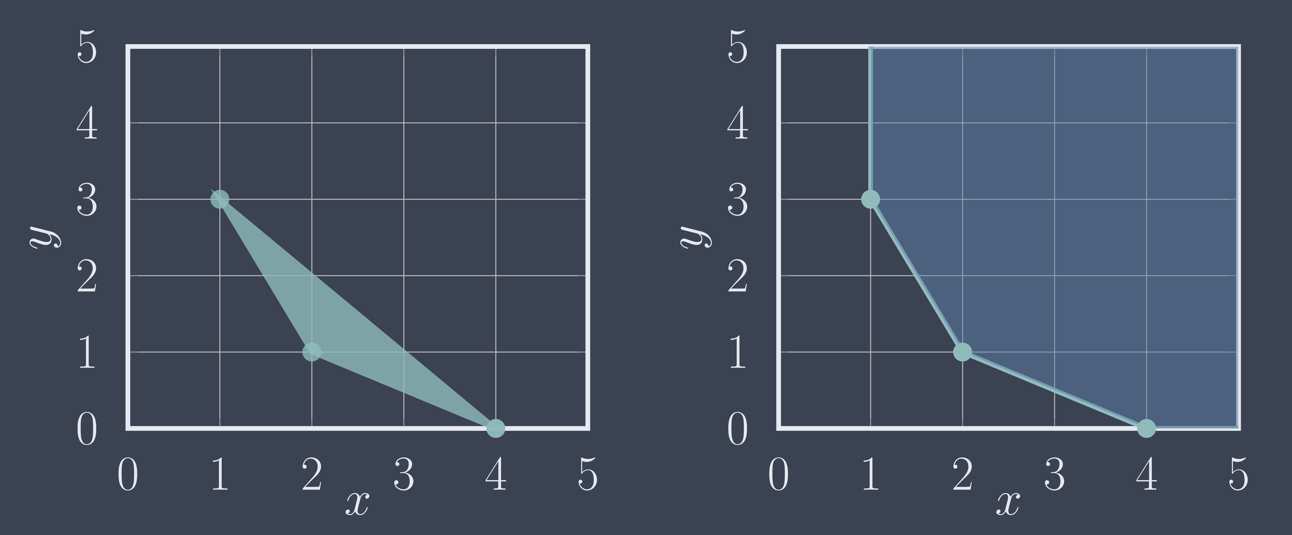

Fig. 1: Left: Newton Polytope, Right: Newton Polyhedron

Rational Powers

What is a Rational Power?

A *rational power* of an ideal, $\overline{I^r}$, is the ideal generated by the lattice points contained in $r \cdot NP(I)$. Figure 2 shows $r \cdot NP(I)$ when $I = (x^4, x^2y, xy^3)$ and $r = \frac{1}{2}$.

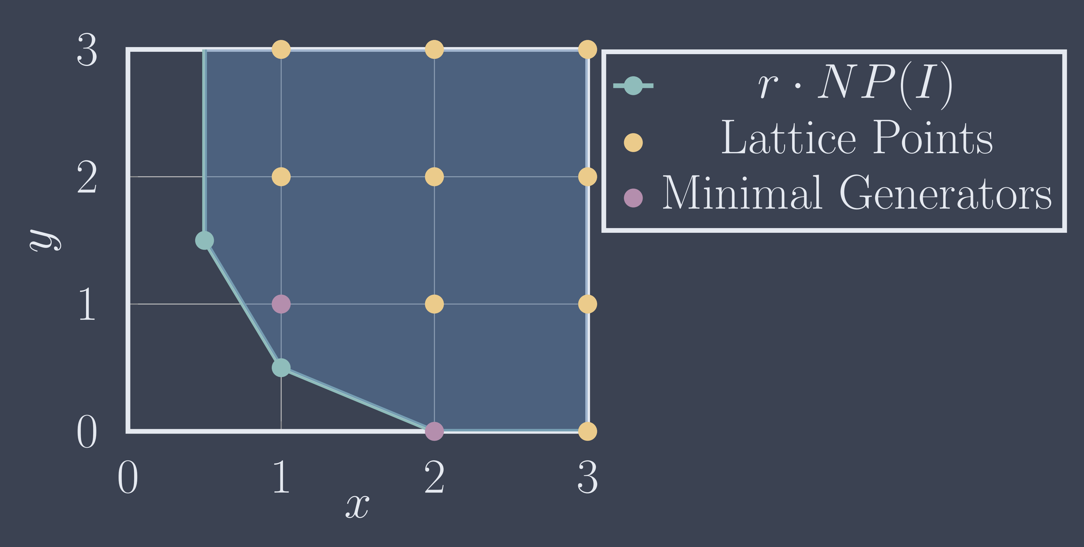

Fig. 2: $\frac{1}{2} \cdot NP(I)$

Looking at the magenta lattice points, we observe that

$$\overline{I^r} = \overline{(x^4, x^2y, xy^3)^{\frac{1}{2}}} = (xy, x^2).$$

Minkowski Algorithm for Computing Rational Powers [1]

We found that the minimal generators of $\overline{I^r}$ are within a predetermined bounded distance of $r \cdot np(I)$. With that, we designed an algorithm which computes the rational powers of ideals. Figure 3 highlights the bounded region in green, in which the minimal generators will be contained.

Outline of the Minkowski Algorithm:

1. Find all lattice points within the predetermined bounded distance of $r \cdot np(I)$.
2. Compute the ideal generated by these points, giving us $\overline{I^r}$.

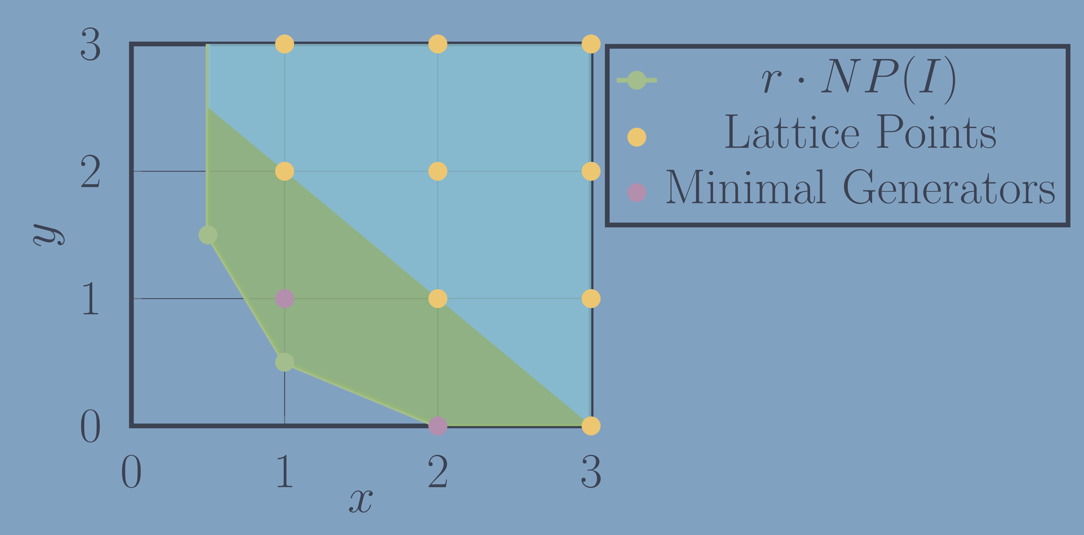

Fig. 3: Minkowski Algorithm example

Jumping Numbers

What is a Jumping Number?

We say that r is a *jumping number* of I if $\overline{I^r} \neq \overline{I^{r+\varepsilon}}$ for all $\varepsilon > 0$.

Figure 4 shows $r \cdot NP(I)$ where $I = (x^4, x^2y, xy^3)$, with $r = \frac{1}{2}$ and $\frac{5}{8}$.

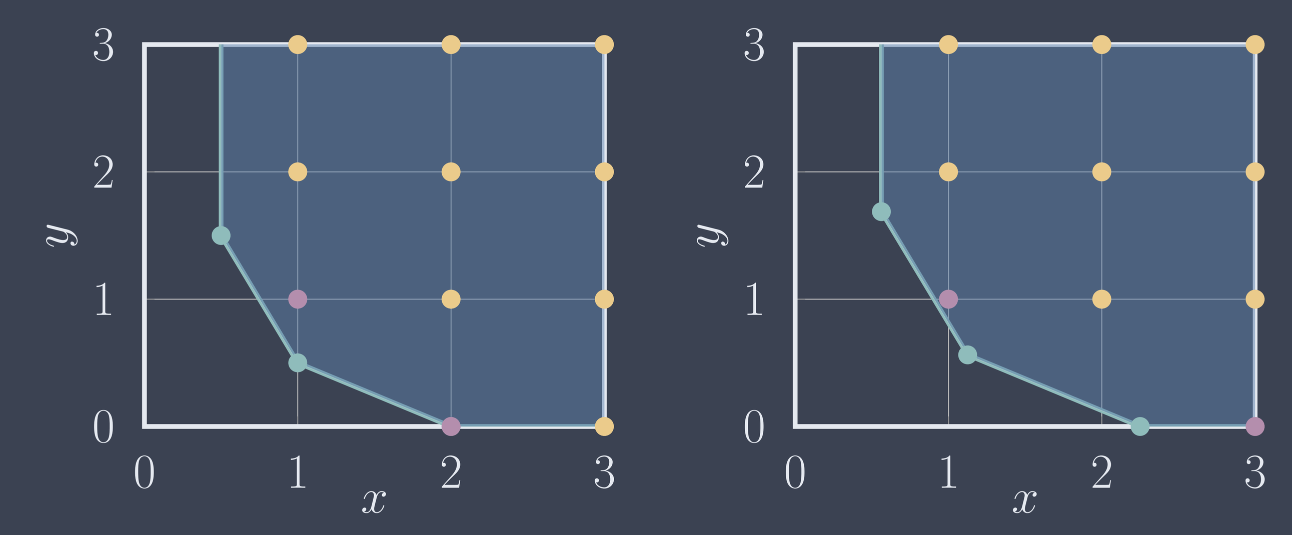

Fig. 4: Left: $r = \frac{1}{2}$, Right: $r = \frac{5}{8}$

Note that when $r = \frac{1}{2}$, increasing r by a small amount will exclude the point $(2, 0)$ from $r \cdot NP(I)$. This removes a minimal generator from $r \cdot NP(I)$, making $r = \frac{1}{2}$ a jumping number of I.

Results [1]

- Jumping numbers correspond to integer solutions to a bounding equation of $r \cdot NP(I)$.
- If r is a jumping number of I then nr is also a jumping number of I for all $n \in \mathbb{N}$.
- All jumping numbers are rational.
- All positive rational numbers are jumping numbers.

Acknowledgments and References

This work was done in collaboration with the 2020 Polymath REU. All thanks to the program and our research mentor Alexandra Seceleanu.

References

[1] Polymath 2020. Rational Powers of Monomial Ideals. in preparation.