# Real Powers of Monomial Ideals

by Josiah Lim, Ethan Roy and Ethan Partida
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# » Outline

* Background
* Geometric Representations
* Real Powers
* Jumping numbers

## Background

* What is a Monomial?
* What is a Monomial Ideal?

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# What is a monomial?

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**Non-examples**

$x + y$ and $xy - z$ are polynomials, not monomials.

## Background

* What is a Monomial?
* What is a Monomial Ideal?

# » What is a Monomial Ideal?

**Definition (Monomial Ideal)**

Let $M = \{\mathtt{m}_1, \ldots, \mathtt{m}_k\}$ be a set of monomials. The ideal generated by $M$, written $I = (\mathtt{m}_1, \ldots, \mathtt{m}_k)$, is the set containing all polynomials which have the form $p_1\mathtt{m}_1 + \ldots + p_k\mathtt{m}_k$ where each $p_i$ is a polynomial.

## Geometric Representations

* Monomials and Monomial Ideals $\longrightarrow$ Lattice Points
* What is a Newton Polytope?
* What is a Newton Polyhedron?

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**Example**

In the $xy$ plane,

$$xy^3 \to (1,3)$$
$$x^2y \to (2,1)$$
$$x^4 \to (4,0)$$

## » Monomial Ideals ⟶ Lattice Points

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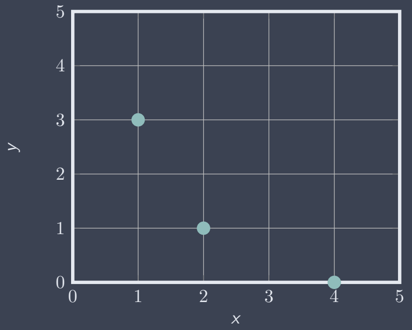
# » Monomial Ideals ⟶ Lattice Points

- Monomials ideals may seem complicated, but pictures are not!
- For the ideal $I = (xy^3, x^2y, x^4)$, the generators are $(1,3)$, $(2,1)$ and $(4,0)$.

Generators of the ideal $I = (xy^3, x^2y, x^4)$



## Geometric Representations

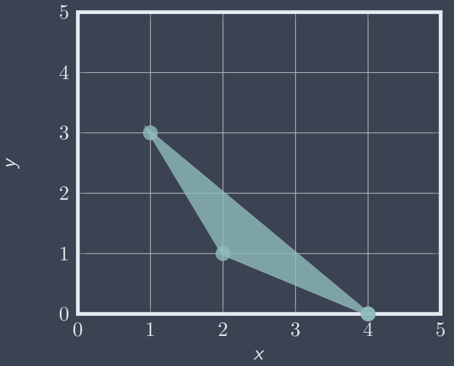
* Monomials and Monomial Ideals $\longrightarrow$ Lattice Points
* What is a Newton Polytope?
* What is a Newton Polyhedron?

# » What is a Newton Polytope?

**Definition (Newton Polytope)**

The newton polytope of an ideal $I$, $np(I)$, is the convex hull of the generators of $I$. ("rubber band around the points")

Example: Newton Polytope of $I = (xy^3, x^2y, x^4)$



## Geometric Representations

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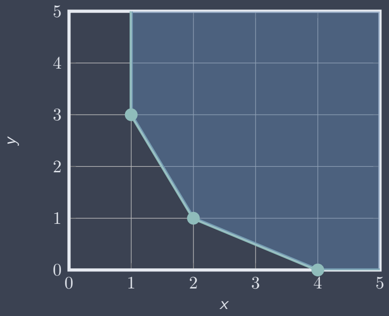
# » What is a Newton Polyhedron?

**Definition (Newton Polyhedron)**

The newton polyhedron of an ideal $I$, $NP(I)$, is the convex hull of all monomials contained in $I$. ("everything up and right of the newton polytope")

Newton Polyhedron of $I = (xy^3, x^2y, x^4)$



## Real Powers

* What is a Real Power?
* Computing Real Powers

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# What is a Real Power?

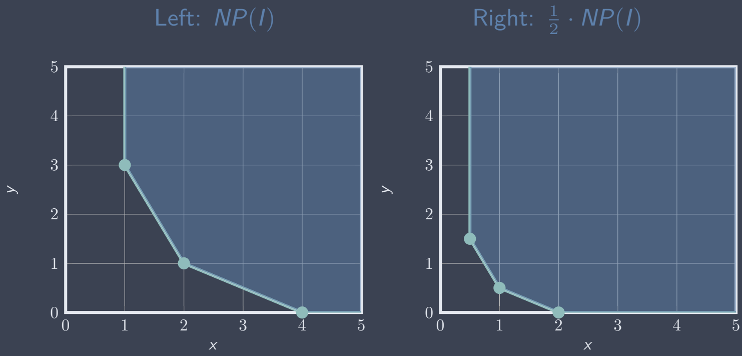
**Definition (Real Power)**

The real power $r$ of an ideal $I$, $\overline{I^r}$, is the ideal generated by the lattice points contained in $r \cdot NP(I)$.

# » Example of Real Power

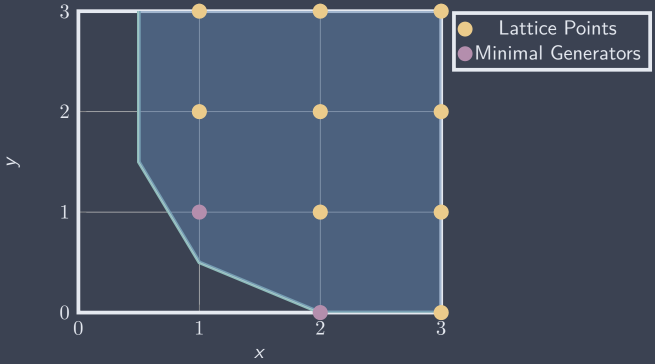
Let $I = (xy^3, x^2y, x^4)$ and $r = \frac{1}{2}$.

To compute $\overline{I^{\frac{1}{2}}}$, we first find $\frac{1}{2} \cdot NP(I)$.



# » Example of Real Power

By looking at the lattice points, we find that $\overline{I^{\frac{1}{2}}} = \overline{(x^4, x^2y, xy^3)^{\frac{1}{2}}} = (xy, x^2)$.



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1. Not much is known about real powers
2. Looking for Patterns
3. Patterns require lots of examples
4. Examples are hard to compute
   * computer program FASTER than working it out by hand

# Minkowski Algorithm

**Theorem (Minkowski Algorithm, loose version)**

The minimal generators of $\overline{I^r}$ are within a predetermined bounded distance of $r \cdot np(I)$.

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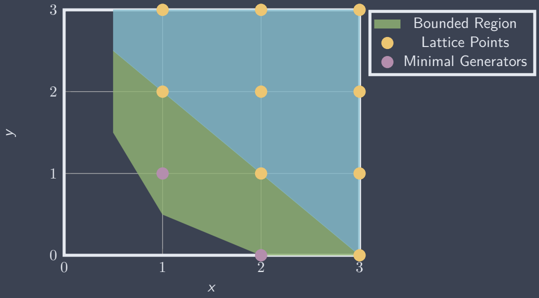
Algorithm Steps:

1. Minkowski sum allows us to find all points within this bounded distance of $r \cdot np(I)$
2. We then compute the ideal generated by these points, this is $\overline{I^r}$.



# Jumping Numbers

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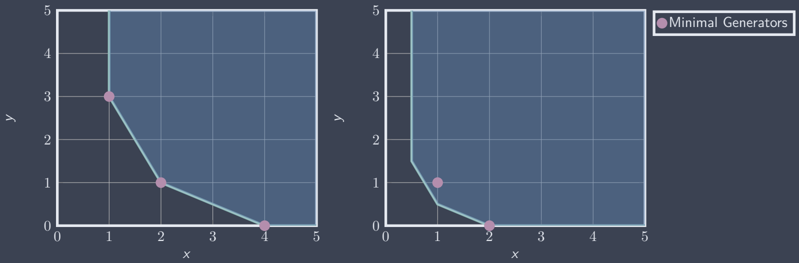
Left: $NP(I)$ Right: $\frac{1}{2} \cdot NP(I)$



# » What is a jumping number?

**Definition (Jumping Number)**

We say that a number $r$ is a jumping number if $\overline{I^r} \neq \overline{I^{r+\epsilon}}$ for all $\epsilon > 0$.

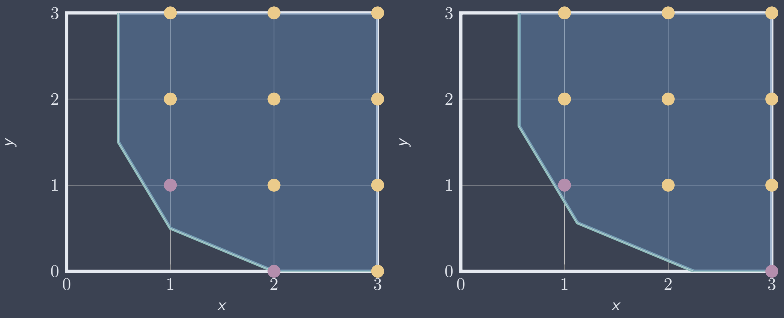
For $I = (x^4, x^2y, xy^3)$, we have that

* $\frac{1}{2}$ is a jumping number
* $\frac{1}{3}$ is not a jumping number

# » $\frac{1}{2}$ is a jumping number

Increasing $\frac{1}{2}$ just a little bit will no longer include the point $(2, 0)$. This removes a minimal generator and changes the ideal. Thus $\frac{1}{2}$ is a jumping number.

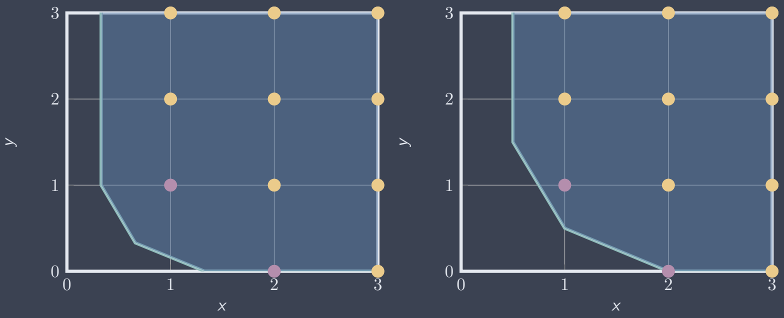
$r = \frac{1}{2}$ vs $r = \frac{5}{8}$



## » $\frac{1}{3}$ is not a jumping number

* By looking at $r \cdot NP(I)$ we can determine $\overline{I^r}$
* Thus we can see, $\overline{I^{\frac{1}{2}}} = \overline{I^{\frac{1}{3}}}$

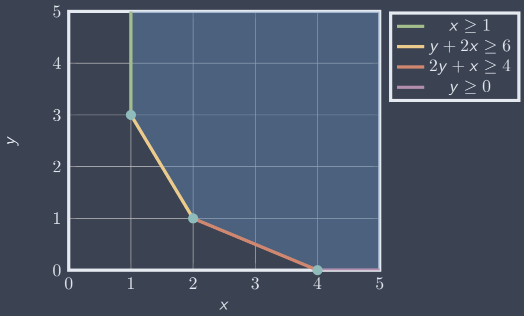
$r = \frac{1}{3}$ vs $r = \frac{1}{2}$



# » A new perspective

We can describe newton polyhedron by a system of linear inequalities.

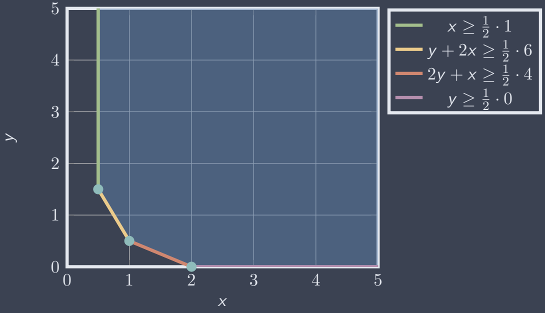
Newton Polyhedron of $(xy^3, x^2y, x^4)$



# A new perspective

Scaling a newton polyhedron corresponds to scaling constants in our inequalities.

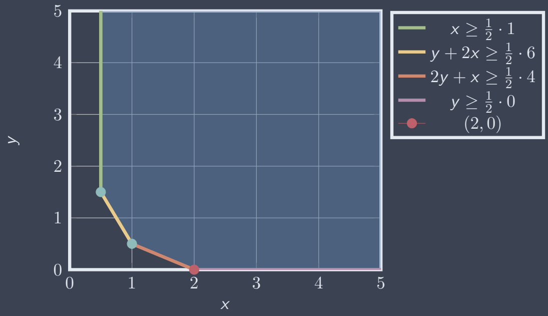
Newton Polyhedron of $\overline{(xy^3, x^2y, x^4)^{\frac{1}{2}}}$



# » A new perspective

Notice that $(2, 0)$ gives equality in $2y + x \geq \frac{1}{2} \cdot 4$ and inequality for the rest of our bounding equations.

Newton Polyhedron of $\overline{(xy^3, x^2y, x^4)^{\frac{1}{2}}}$



# » A new perspective

**Theorem**

$r \in \mathbb{R}$ is a jumping number for a monomial ideal $I$ if and only if there is an integer solution to the (interesting) bounding inequalities of $r \cdot NP(I)$.

# » Corollaries

Using this reformulation, we were able to prove many interesting corollaries:

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* If $r$ is a jumping number of $I$ then $nr$ is also a jumping number of $I$ for all $n \in \mathbb{N}$.

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* If $r$ is a jumping number of $I$ then $nr$ is also a jumping number of $I$ for all $n \in \mathbb{N}$.
* If v is a vertex of $NP(I)$, then for all $n \in \mathbb{N}$ the number $r_n = \frac{n}{\gcd(v_1, \cdots, v_d)}$ is a jumping number of $I$ .

# » Thank You!

Any Questions?